Comparative study of the numerical convergence for different sets of basis functions used in the moment method analysis of rectangular patch antenna

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ABSTRACT

The selection of basis functions in the moment method analysis of rectangular patch antenna plays an important role in determining the rate of convergence of the numerical results, and that an improper choice can lead to erroneous results. In this work, three types of basis functions are used to expand the unknown surface current distribution on the rectangular patch. The first type of basis functions is formed by the set of TM modes of a rectangular cavity with magnetic side walls. The second type of basis functions employs sinusoidal functions with the proper edge singularity. The third type of expansion basis functions consists of combinations of Chebyshev polynomials, with weighting factors to incorporate the edge condition. The numerical convergence for these three types of basis functions is discussed in detail. The advantages and disadvantages brought by the use of each type of these basis functions are also given. **KEY WORDS: rectangular patch; Galerkin's method; basis functions; numerical convergence.**

1. INTRODUCTION

The electric field integral equation method implemented in the Fourier transform domain has been extensively used in the last two decades as an efficient way to predict the electromagnetic behaviours of the rectangular microstrip patch antenna. The Galerkin method is considered to be the standard procedure for solving this class of integral equations. It gives the fundamental quantity of interest, namely the electric current distribution on the patch surface from which all the other required antenna parameters can be obtained [1]. In the literature, different types of basis functions have been successfully used to expand the unknown patch current. However, the questions pertaining to the choice of the basis functions (modes) for each type of expansion basis functions have not been addressed in much detail, except in the context of convergence of the integrals involved in the application of the moment method both in the spatial and spectral domain approaches and strictly from a mathematical point of view [2]. In this work, three types of basis functions are used to expand the unknown surface current distribution on the rectangular patch. The first type of basis functions is formed by the set of TM modes of a rectangular cavity with magnetic side walls. The second type of basis functions employs sinusoidal functions with the proper edge singularity. The third type of expansion basis functions consists of combinations of Chebyshev polynomials, with weighting factors to incorporate the edge condition. The numerical convergence for these three types of basis functions is discussed in detail.

We begin first by discussing briefly, in Section 2, the application of Galerkin's method in the Fourier transform domain to the full-wave computation of the complex resonant frequencies of rectangular microstrip patches. The selection of basis functions is discussed in Section 3. The numerical convergence using the three types of basis functions, mentioned above, is discussed in Section 4. The advantages and disadvantages brought by the use of each type of these basis functions are also given. Concluding remarks are summarized in Section 5.

2. FORMULATION OF THE PROBLEM

Considered here is a rectangular microstrip patch with dimensions (a, b) along the two axes (x, y), respectively. Let the thickness and the relative permittivity of the substrate be denoted by d_1 and ε_{r1}, respectively. In the Fourier transform domain, the tangential electric field on the plane of the patch due to the patch currents J_x and J_y can be written, in terms of the electric field Green's function, as follows:

$$\begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \cdot \begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} \tag{1}$$

where the Green's function Q_{ij} is the contribution of a *j*-directed electric current element at the microstrip to the electric field E_i at the microstrip plane. Note that the electric field Green's functions in the spectral domain can be easily obtained. To solve for the surface current density on the patch by the method of moment, the first step is to expand the surface current densities by a linear combination of the expansion functions as follows:

$$J_x = \sum_{n=1}^{N} a_n J_{xn} \tag{2a}$$

$$J_y = \sum_{m=1}^{M} b_m J_{ym} \tag{2b}$$

where a_n and b_m are the unknown coefficients of the expansion functions J_{xn} and J_{ym}, respectively. Substituting the Fourier transform of equation (2) into equation (1) followed by testing with the same set of basis functions that was used in the expansion of the patch current, one arrives at the following matrix equation:

$$\begin{bmatrix} \left(\bar{Z}^{11}\right)_{N\times N} & \left(\bar{Z}^{12}\right)_{N\times M} \\ \left(\bar{Z}^{21}\right)_{M\times N} & \left(\bar{Z}^{22}\right)_{M\times M} \end{bmatrix} \cdot \begin{bmatrix} (a)_{N\times 1} \\ (b)_{M\times 1} \end{bmatrix} = 0 \tag{3}$$

where

$$\left(\bar{Z}^{11}\right)_{qn} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_{xx} \tilde{J}_{xq}(-k_s).\tilde{J}_{xn}(k_s) dk_x dk_y \tag{4a}$$

$$\left(\bar{Z}^{12}\right)_{qm} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_{xy} \tilde{J}_{xq}(-k_s).\tilde{J}_{ym}(k_s) dk_x dk_y \tag{4b}$$

$$\left(\bar{Z}^{21}\right)_{ln} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_{yx} \tilde{J}_{yl}(-k_s).\tilde{J}_{xn}(k_s) dk_x dk_y \tag{4c}$$

$$\left(\bar{Z}^{22}\right)_{lm} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q_{yy} \tilde{J}_{yl}(-k_s).\tilde{J}_{ym}(k_s) dk_x dk_y \tag{4d}$$

For the existence of a non-trivial solution of (3), we must have

$$\det [\bar{\mathbf{Z}}(f)] = 0 \tag{5}$$

where $\bar{\mathbf{Z}}$ is the entire matrix in (3). Muller's method which involves three initial guesses is used for root seeking of (5) to obtain the operating frequency and the quality factor of the antenna.

3. CHOICE OF BASIS FUNCTIONS

When using the method of moments to solve electromagnetic problems, a crucial factor is the appropriate choice of basis functions [3]. In this work, three types of basis functions are used to expand the unknown surface current distribution on the rectangular patch. The first type of basis functions is formed by the set of TM modes of a rectangular cavity with magnetic side walls. These current modes are given by

$$J_{xn}(x,y) = \sin\left[\frac{n_1\pi}{a}(x+\frac{a}{2})\right]\cos\left[\frac{n_2\pi}{b}(y+\frac{b}{2})\right] \quad (6a)$$

$$J_{ym}(x,y) = \sin\left[\frac{m_2\pi}{b}(y+\frac{b}{2})\right]\cos\left[\frac{m_1\pi}{a}(x+\frac{a}{2})\right] \quad (6b)$$

The second type of basis functions employs sinusoidal functions with the proper edge singularity. These basis functions are given by

$$J_{xn}(x,y) = \frac{1}{\sqrt{(b/2)^2 - y^2}}\sin\left[\frac{n_1\pi}{a}(x+\frac{a}{2})\right]\cos\left[\frac{n_2\pi}{b}(y+\frac{b}{2})\right] \quad (7a)$$

$$J_{ym}(x,y) = \frac{1}{\sqrt{(a/2)^2 - x^2}}\sin\left[\frac{m_2\pi}{b}(y+\frac{b}{2})\right]\cos\left[\frac{m_1\pi}{a}(x+\frac{a}{2})\right] \quad (7b)$$

The third type of expansion basis functions consists of combinations of Chebyshev polynomials, with weighting factors to incorporate the edge condition. These basis functions are given by

$$J_{xn}(x,y) = \sqrt{\frac{1-(2x/a)^2}{1-(2y/b)^2}}.U_{n_1}(2x/a).T_{n_2}(2y/b) \quad (8a)$$

$$J_{ym}(x,y) = \sqrt{\frac{1-(2y/b)^2}{1-(2x/a)^2}}.U_{m_2}(2y/b).T_{m_1}(2x/a) \quad (8b)$$

where $T_n(\cdot)$ and $U_n(\cdot)$ are the nth-order Chebyshev polynomials of the first and second kind, respectively [4].

4. NUMERICAL RESULTS AND DISCUSSION

In this Section, the numerical convergence of our calculated resonant frequencies and quality factors, using the three types of basis functions mentioned in Section 3, are studied. The considered mode is the TM_{01} mode. The patch dimension is $a \times b = 1.9\text{ cm} \times 2.29\text{cm}$ and the substrate has a relative permittivity ε_{r1}= 2.32 and thickness $d_1 = 1.59\text{ mm}$.

In Table I, the convergence of the numerical results using cavity modes basis functions are investigated with respect to the number of basis functions. The results indicate that the choice of the couples (n_1, n_2) and (m_1, m_2) should not be done in an arbitrary way, but it is subjected to the two following criteria:

- ✓ n_1 and m_1 must be both even.
- ✓ n_2 and m_2 must be both odd.

If we use an additional couple which does not check the conditions above, the size of the matrix impedance increases without improvement of convergence. Note that the study of convergence reported in the open literature [5, 6] has not been done in a suitable way, since these two conditions have not been respected during the expansion of the current in series of basis functions. The numerical results also indicate that only one basis function in the y direction suffices to obtain the resonant frequency with acceptable accuracy. The theoretical frequency obtained with $(N=0, M=1)$ agrees very well with the measured data given in [7] with a small shift of 0.019 GHz . The advantages brought by the use of one basis function are

- ✓ A considerable saving of the computation time because only one term in the impedance matrix is to be evaluated.
- ✓ Muller's method is shown to converge to the correct frequency for a large choice of initial guesses.

In Table II, the convergence of the numerical results using sinusoidal basis functions with edge singularity are studied with respect to the number of basis functions. It is seen that the conditions imposed on the choice of the sinusoidal basis functions with edge singularity are similar to those imposed on the selection of the sinusoidal basis functions without edge singularity. Now, one basis function is unable to ensure the convergence of the numerical results. An additional mode in the y direction being necessary. This last is defined by the couple $(2,1)$. Note that the computation time required to obtain the complex resonant frequency when sinusoidal basis functions with edge singularity are used is longer than that necessary for the calculation of the complex resonant frequency when sinusoidal basis functions without edge singularity are used for the following reasons:

- ✓ Three elements in the impedance matrix are to be evaluated.
- ✓ The length of the integration path required to reach numerical convergence when the sinusoidal basis functions with edge singularity are used is 250 k_0, while the semi-infinite integral is truncated at an upper bound of 60 k0 when the sinusoidal basis functions without edge singularity are used.
- ✓ The Fourier transform of the sinusoidal basis functions without edge singularity is expressed in term of the cardinal sine function (see Appendix A), while the Fourier transform of the sinusoidal basis functions with edge singularity is expressed in term of the Bessel function of the first kind of order zero (see Appendix A). It is well known that the numerical computation of Bessel functions is time-consuming.

(see Appendix A).

5. CONCLUSIONS

Three sets of basis functions have been used to expand the unknown surface current distribution on the rectangular patch. The numerical convergence for these three sets of basis functions has been discussed in detail. The results obtained have indicated that it is not necessary to consider the edge singularity to obtain fast numerical convergence of the complex resonant frequency for a rectangular microstrip structure. For each set of basis functions, we have shown that the choice of the modes should not be done in an arbitrary way, but it is subjected to some criteria. These criteria have been given for each set of basis functions. Through a thorough examination of the convergence question, we conclude that the set of basis functions issued from the magnetic wall cavity model are the best set of basis functions to be used in the moment method analysis of rectangular patch antenna because it ensure rapid convergence of the Galerkin method with a good exactitude of the results.

APPENDIX A

This appendix regroups details of calculation of the Fourier transforms of the three types of basis functions used in the approximation of the unknown current on the rectangular patch. The Fourier transforms of J_{xn} and J_{ym} are

$$\tilde{J}_{xn} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J_{xn} \exp(-\mathrm{i}\, k_x x - \mathrm{i}\, k_y y)\, dx\, dy \qquad \text{(A.1a)}$$

$$\tilde{J}_{ym} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} J_{ym} \exp(-\mathrm{i}\, k_x x - \mathrm{i}\, k_y y)\, dx\, dy \qquad \text{(A.1b)}$$

A.1. Cavity modes basis functions

Substituting equation (6a) into equation (A.1a) and equation (6b) into equation (A.1b), and using Moivre's rule, we obtain the following expressions for $\tilde{J}_{xn}$ and $\tilde{J}_{ym}$:

$$\tilde{J}_{xn} = \tilde{I}_{xx}(k_x) \cdot \tilde{I}_{xy}(k_y) \qquad \text{(A.2a)}$$

$$\tilde{J}_{ym} = \tilde{I}_{yx}(k_x) \cdot \tilde{I}_{yy}(k_y) \qquad \text{(A.2b)}$$

with

$$\tilde{I}_{xx}=\frac{\mathrm{i}\,a}{2}\left[\exp(-\mathrm{i}n_1\pi/2)\cdot\mathrm{sinc}(k_x a/2+n_1\pi/2)-\exp(\mathrm{i}n_1\pi/2)\cdot\mathrm{sinc}(k_x a/2-n_1\pi/2)\right] \quad \text{(A.3a)}$$

$$\tilde{I}_{xy}=\frac{b}{2}\left[\exp(-\mathrm{i}n_2\pi/2)\cdot\mathrm{sinc}(k_y b/2+n_2\pi/2)+\exp(\mathrm{i}n_2\pi/2)\cdot\mathrm{sinc}(k_y b/2-n_2\pi/2)\right] \quad \text{(A.3b)}$$

$$\tilde{I}_{yx}=\frac{a}{2}\left[\exp(-\mathrm{i}m_1\pi/2)\cdot\mathrm{sinc}(k_x a/2+m_1\pi/2)+\exp(\mathrm{i}m_1\pi/2)\cdot\mathrm{sinc}(k_x a/2-m_1\pi/2)\right] \quad \text{(A.4a)}$$

$$\tilde{I}_{yy}=\frac{\mathrm{i}b}{2}\left[\exp(-\mathrm{i}m_2\pi/2)\cdot\mathrm{sinc}(k_y b/2+m_2\pi/2)-\exp(\mathrm{i}m_2\pi/2)\cdot\mathrm{sinc}(k_y b/2-m_2\pi/2)\right] \quad \text{(A.4b)}$$

A.2. Sinusoidal basis functions with edge singularity

Substituting equation (7a) into equation (A.1a) and equation (7b) into equation (A.1b), and using Moivre's rule in conjunction with the following integral [8]:

$$\int_{-w}^{+w}\frac{\exp(i\xi x)}{\sqrt{1-(x/w)^2}}dx=\pi w J_0(w\xi) \quad \text{(A.5)}$$

we obtain the following expressions for J_{xk} and J_{ym}

$$\tilde{J}_{xn}=\tilde{I}_{xx}(k_x).\tilde{I}_{xy}(k_y) \quad \text{(A.6a)}$$

$$\tilde{J}_{ym}=\tilde{I}_{yx}(k_x).\tilde{I}_{yy}(k_y) \quad \text{(A.6b)}$$

where $\tilde{I}_{xx}$ and $\tilde{I}_{yy}$ are similar to those defined in equations (A.3a) and (A.4b), and

$$\tilde{I}_{xy}=\frac{(-i)^{n_2}\pi}{2}\left[J_0(k_y b/2+n_2\pi/2)+(-1)^{n_2}J_0(k_y b/2-n_2\pi/2)\right] \quad \text{(A.7a)}$$

$$\tilde{I}_{yx}=\frac{(-i)^{m_1}\pi}{2}\left[J_0(k_x a/2+m_1\pi/2)+(-1)^{m_1}J_0(k_x a/2-m_1\pi/2)\right] \quad \text{(A.7b)}$$

In equations (A.5), (A.7a), and (A.7b), $J_0(.)$ is the Bessel function of the first kind of order zero.

A.3. Chebyshev polynomials basis functions with edge condition

Substituting equation (8a) into equation (A.1a) and equation (8b) into equation (A.1b), and using the following integrals [8]:

$$\int_{-w}^{+w}\frac{T_n(x/w)\exp(i\xi x)}{\sqrt{1-(x/w)^2}}dx=\pi w i^n J_n(w\xi) \quad \text{(A.8a)}$$

$$\int_{-w}^{+w} \sqrt{1-(x/w)^2}\,U_n(x/w)\exp(i\xi x)dx = \pi i^n \frac{n+1}{\xi} J_{n+1}(w\xi) \qquad \text{(A.8b)}$$

we obtain the following expressions for $\tilde{J}_{xn}$ and $\tilde{J}_{ym}$

$$\tilde{J}_{xn} = \pi^2 \frac{ab(-i)^{n_1+n_2}}{4} . \frac{n_1+1}{k_x a/2} J_{n_1+1}(k_x a/2) J_{n_2}(k_y b/2) \qquad \text{(A.9a)}$$

$$\tilde{J}_{ym} = \pi^2 \frac{ab(-i)^{m_1+m_2}}{4} . \frac{m_2+1}{k_y b/2} J_{m_2+1}(k_y b/2) J_{m_1}(k_x a/2) \qquad \text{(A.9b)}$$

Using the following property for the Bessel function of the first kind:

$$\frac{nJ_n(x)}{x} = \frac{1}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right] \qquad \text{(A.10)}$$

equations (A.9a) and (A.9b) become

$$\tilde{J}_{xn} = \pi^2 \frac{ab(-i)^{n_1+n_2}}{8}\left[J_{n_1}(k_x a/2) + J_{n_1+2}(k_x a/2)\right] J_{n_2}(k_y b/2) \qquad \text{(A.11a)}$$

$$\tilde{J}_{ym} = \pi^2 \frac{ab(-i)^{m_1+m_2}}{8}\left[J_{m_2}(k_y b/2) + J_{m_2+2}(k_y b/2)\right] J_{m_1}(k_x a/2) \qquad \text{(A.11b)}$$

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Table I.

Convergence pattern of the resonant frequency and quality factor with respect to the number of cavity modes basis functions;

$a \times b = 1.9$ cm $\times$ 2.29 cm , $\varepsilon_{r1} = 2.32$, d_1=1.59 mm .

Modes used in the x direction		Modes used in the y direction		Resonant frequency (GHz)	Quality factor
N	(n_1, n_2)	M	(m_1, m_2)		
0		1	(0,1)	4.1231	36.346
5	(1,0), (1,1), (1,2), (2,0), (2,2)	1	(0,1)	4.1231	36.346
1	(2,1)	1	(0,1)	4.1087	36.762
0		5	(0,1), (0,2), (1,1), (1,2), (2,2)	4.1231	36.346
0		2	(0,1), (2,1)	4.1195	37.333
5	(1,0), (1,1), (1,2), (2,0), (2,2)	5	(0,1), (0,2), (1,1), (1,2), (2,2)	4.1231	36.346
1	(2,1)	2	(0,1), (2,1)	4.1336	35.792
2	(2,1), (2,3)	4	(0,1), (2,1), (0,3), (2,3)	4.1151	35.824
4	(2,1), (2,3), (4,1), (4,3)	6	(0,1), (2,1), (0,3), (2,3), (4,1), (4,3)	4.1181	35.579

Table II.

Convergence pattern of the resonant frequency and quality factor with respect to the number of sinusoidal basis functions with edge singularity;

$a \times b = 1.9$ cm $\times$ 2.29 cm , $\varepsilon_{r1} = 2.32$, d_1=1.59 mm .

Modes used in the x direction		Modes used in the y direction		Resonant frequency (GHz)	Quality factor
N	(n_1, n_2)	M	(m_1, m_2)		
0		1	(0,1)	4.1842	36.878
5	(1,0), (1,1), (1,2), (2,0), (2,2)	1	(0,1)	4.1842	36.878
1	(2,1)	1	(0,1)	4.0821	39.426
0		5	(0,1), (0,2), (1,1), (1,2), (2,2)	4.1842	36.878
0		2	(0,1), (2,1)	4.1163	36.840
5	(1,0), (1,1), (1,2), (2,0), (2,2)	5	(0,1), (0,2), (1,1), (1,2), (2,2)	4.1842	36.878
1	(2,1)	2	(0,1), (2,1)	4.1323	35.742
2	(2,1), (2,3)	4	(0,1), (2,1), (0,3), (2,3)	4.1090	35.533
4	(2,1), (2,3), (4,1), (4,3)	6	(0,1), (2,1), (0,3), (2,3), (4,1), (4,3)	4.1042	35.564